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Perturbations of supinf problems with constraints

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Abstract

We investigate constrained supinf problems for functions of two variables. Conditions are given which assure that the objective function can be perturbed by continuous functions with arbitrary small norms in such a way that the supinf problem for the perturbed function has a solution. We also give a characterization of the notion of well-posedness for such problems.

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1 Introduction

In this article we consider the following supinf problem with constraints:

$$(P) \qquad \sup_{x\in X} \inf_{y\in Kx} f(x,y),$$

where X and Y are (everywhere in this article) completely regular topological spaces, $f : X\times Y \to [-\infty,+\infty]$ is an extended real-valued function and $K : X \rightrightarrows Y$ is a set-valued mapping with nonempty images. A solution to the problem (P) is every couple $(x_0, y_0) \in X \times Y$ such that $y_0 \in Kx_0$ and

$$f(x_0, y_0) = \inf_{y\in Kx_0} f(x_0, y) = \sup_{x\in X} \inf_{y\in Kx} f(x, y).$$

The setting of this problem includes, for example, "leader-follower" games. In such a game the first player (who is considered as a leader) makes its choice first with the aim to maximize her/his profit which is given by the function f. After choosing $x \in X$, the second player makes her/his choice in the set of feasible choices Kx. The gain for the first player is $f(x,y)$, which depends on the choice of the second player, and *the value of the problem*

$$v_f := \sup_{x\in X} \inf_{y\in Kx} f(x,y)$$

expresses the guaranteed utility which the first player can assure even with the worst behaviour of the second one. In the particular case when the set-valued mapping K is given as the set of solutions to some optimization problem, i.e. for any $x \in X$ we have

$$Kx := \{y' : g(x,y') = \inf_{y\in Y} g(x,y)\},$$

where $g : X \times Y \to \mathbb{R}$ is a given function, this game is known as a Stackelberg problem (sometimes the latter problem is also called "two level optimization problem"). For more information about the Stackelberg problem see, e.g., [9], [6], [7].

When we have $Kx = Y$ for any $x \in X$, the problem (P) gives, for example, the guaranteed gain for some of the two players in noncooperative games, or can be considered also from the point of view of the study of the existence of saddle points for the function f.

Let us come back to the general problem (P) with constraints. Evidently, without additional assumptions on the function f, on the spaces X and Y and on the set-valued mapping K, the problem (P) may not have a solution. The principal aim of the paper is to find conditions which allow perturbations of the original function f by suitable continuous functions in such a way that the perturbed function has a solution for the problem (P). The validity of such kind of result is usually known in optimization as variational principle for the corresponding class of problems. Such results for functions of two variables have been obtained by McLinden in [8] (using Ekeland variational principle [1]) and for the case of unconstrained supinf problems (i.e. in which $Kx = Y$ for each $x \in X$) by Kenderov et al. in [5]. A variational principle for several classes of usual optimization problems with constraints was obtained by Ioffe et al. in [2].

We will also be interested in investigating stronger notions than merely the existence of solutions to supinf problems. More precisely, we will be interested in the well-posedness of the corresponding problem. To introduce this notion for supinf problems, let us first remind that the minimization problem determined by a given extended real-valued function $h : Z \to \mathbb{R} \cup \{+\infty\}$ defined in a topological space Z is called *Tykhonov well-posed* if h has a unique minimum point $z_0 \in Z$ and, moreover for every minimizing sequence $\{z_n\}_{n=1}^{\infty} \subset Z$ for h, i.e. $h(z_n) \to \inf_Z h$, it follows that $z_n \to z_0$. An equivalent definition of the Tykhonov well-posedness is: *every minimizing sequence for* h *converges to some minimizer of* h. Analogously, for a given $h : Z \to \mathbb{R} \cup \{-\infty\}$ the corresponding maximization problem is Tykhonov well-posed if there is a unique maximizer of h in Z towards which converges every maximizing sequence for the function h (equivalently, if the minimization problem for $-h$ is well-posed in the sense of Tykhonov).

The first notion of well-posedness for the supinf problem (P) concerns only the first player. Having in mind the leader-follower game, a solution for the leader player (called *sup-solution*) is any point $x_0 \in X$ such that $v_f = \inf_{y \in Kx_0} f(x_0, y)$. The problem (P) is called *sup-well-posed* if the problem to maximize the function $v(\cdot) := \inf_{y \in K(\cdot)} f(\cdot, y)$ is well-posed in the sense of Tykhonov. In the latter case, obviously, there is only one sup-solution.

We will be also interested in a stronger notion of well-posedness for the supinf problem (P). Namely, (see e.g. [3]) the problem (P) is called *well-posed* if every optimizing sequence for the problem (P) converges to some (in fact, unique) solution (x_0, y_0) of (P) (here, and in the sequel, on $X \times Y$ we consider the product topology, generated by the topologies of X and Y). A sequence $((x_n, y_n))_n \subset X \times Y$ is called *optimizing* for (P) if:

1. $y_n \in Kx_n$ for every n;
2. $v(x_n) \to v_f = \sup_{x \in X} \inf_{y \in Kx} f(x, y)$;
3. $f(x_n, y_n) \to v_f$

It is evident that if (P) is well-posed with unique solution (x_0, y_0), then the problem (P) is sup-well posed with unique sup-solution x_0. The converse is not true, in general. Several (generic) variational principles for the problem (P), concerning well-posedness were obtained by Kenderov and Lucchetti in [3].

2 Preliminaries

Let Z be a completely regular topological space and $h : Z \to \mathbb{R} \cup \{+\infty\}$ be an extended real-valued function. The set $\operatorname{dom}(h) = \{z \in Z : h(z) < +\infty\}$ stands as usual for the *domain* of h. In the case when h takes values in $\mathbb{R} \cup \{-\infty\}$, the set $\operatorname{dom}(h)$ again consists of all points in Z at which h is finite. The function $h : Z \to \mathbb{R} \cup \{+\infty\}$ is called *proper* if its domain is not empty. Denote by $C(Z)$ the space of all continuous, bounded real-valued functions defined on Z. The space $C(Z)$ equipped with the norm of uniform convergence $\|g\|_{Z,\infty} := \sup\{|g(z)| : z \in Z\}$, $g \in C(Z)$, is a real Banach space. Let us first formulate the following result from [4] (see also Remark 2.2 from [5]) which will be used in the sequel ($\mathbb{R}_+$ meaning the set of non-negative reals):

Lemma 1 ([4][Lemma 2.1]) *Let $h : Z \to \mathbb{R}\cup\{+\infty\}$ be a proper lower semicontinuous function which is bounded from below. Let $z_0 \in \operatorname{dom}(h)$ and $\varepsilon > 0$ be such that $h(z_0) < \inf_Z h + \varepsilon$. Then, there exists a continuous bounded function $g : Z \to \mathbb{R}_+$ such that $g(z_0) = 0$, $\|g\|_{Z,\infty} \le \varepsilon$ and the function $h + g$ attains its minimum in Z at z_0. Moreover, g can be chosen such that $\|g\|_{Z,\infty} = h(z_0) - \inf_Z h$.*

The above lemma concerns unconstrained optimization, i.e. when we search for the minimum of h on the whole space X. But in special cases it can be also modified to have perturbations for constrained optimization problems. Namely, the following result is true:

Lemma 2 *Let $h : Z \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function which is bounded from below. Let A be a closed subset of Z such that $A \cap \operatorname{dom}(h) \neq \emptyset$. Let $z_0 \in A \cap \operatorname{dom}(h)$ and $\varepsilon > 0$ be such that $h(z_0) < \inf_A h + \varepsilon$. Then, there exists a continuous bounded function $g : Z \to \mathbb{R}_+$ such that $g(z_0) = 0$, $\|g\|_{Z,\infty} \le \varepsilon$ and the function $h + g$ attains its minimum in A at z_0. Moreover, g can be chosen such that $\|g\|_{Z,\infty} = h(z_0) - \inf_A h$.*

Proof Define $h' : Z \to \mathbb{R} \cup \{+\infty\}$ by $h'(z) := h(z)$ if $z \in A$ and $h'(z) = +\infty$ otherwise. It is not difficult to see that h' is a proper bounded from below lower semicontinuous function (which coincides with h on A) and $\inf_Z h' = \inf_A h' = \inf_A h$. Apply Lemma 1 for the function h'. It is easily seen that the obtained function $g \in C(Z)$, after applying Lemma 1 for the function h', satisfies the conclusions of Lemma 2.

Recall that a (nonempty valued) set-valued mapping $K : X \rightrightarrows Y$ is called lower semicontinuous (lsc) at $x \in X$ if, for every open set $V \subset Y$ such that $Kx \cap V \neq \emptyset$, there is an open neighbourhood U of x such that $Kx' \cap V \neq \emptyset$ for all $x' \in U$. The mapping K is called lower semicontinuous in X if it is lower semicontinuous at each point of X. The mapping $K : X \rightrightarrows Y$ is called upper semicontinuous (usc) at $x \in X$ if, for every open $V \subset Y$ such that $Kx \subset V$, there is an open neighbourhood U of x such that $Kx' \subset V$ for all $x' \in U$. The mapping K is called upper semicontinuous in X if it is upper semicontinuous at each point of X.

The following result is well-known. A sketch of the proof is given for the sake of completeness:

Lemma 3 *Let $f : X \times Y \to [-\infty, +\infty]$ be an extended real-valued function such that f is upper semicontinuous in $X \times Y$. Let $K : X \rightrightarrows Y$ be a lsc set-valued mapping with nonempty images. Then the function $v(\cdot) = \inf_{y\in K(\cdot)} f(\cdot, y)$ is upper semicontinuous in X.*

Proof Let $x_0 \in X$ and suppose that $v(x_0) = \inf_{y\in Kx_0} f(x_0, y) < \infty$ (otherwise we are done). Take any $t \in \mathbb{R}$ with $v(x_0) < t$. Then, for some $y_0 \in Kx_0$ we will have $f(x_0, y_0) < t$. Since f is upper semicontinuous at (x_0, y_0) there are nonempty open neighborhoods U of x_0 and V of y_0 such that $f(x, y) < t$ for each $(x, y) \in U \times V$. Since $y_0 \in Kx_0 \cap V$ and K is lower semicontinuous at x_0, we may think that U is chosen in such a way that $Kx \cap V \neq \emptyset$ for every $x \in U$. Now, if $x \in U$, let $y_x \in Kx \cap V$. Then, we have,

$$v(x) = \inf_{y\in Kx} f(x, y) \le f(x, y_x) < t,$$

which shows that $v(\cdot)$ is upper semicontinuous in X.

3 Perturbations of supinf problems

In this section we turn back to our general question to find conditions under which we can perturb a given function of two variables by continuous functions in such a way that the perturbed function has a solution for the supinf problem (P). We will suppose the following assumptions for the objective function $f : X \times Y \to [-\infty, +\infty]$ and for the set-valued mapping $K : X \rightrightarrows Y$, the last two of which, in the case when $Kx = Y$ for every $x \in X$, have been already considered in [5]:

(A1) the function $f : X \times Y \to [-\infty, +\infty]$ is upper semicontinuous in $X \times Y$;
(A2) the mapping $K : X \rightrightarrows Y$ is with nonempty closed images and is lsc in X;
(A3) the function $v(\cdot) = \inf_{y \in K(\cdot)} f(\cdot, y)$ is bounded above in X and proper as a function with values in $\mathbb{R} \cup \{-\infty\}$;
(A4) for every $x \in X$ the function $f(x, \cdot)$ is lower semicontinuous in Y.

First, we will prove the following basic perturbation result, which is a variational principle for supinf problems with constraints, and which shows that we can make as small as we wish perturbations of a function of two variables satisfying the conditions (A1)-(A4) (together with the constrained mapping K), by a difference of individually continuous functions defined in Y and X respectively, such that the perturbed function has a solution for the supinf problem. Its proof uses the idea of the proof of Proposition 2.6 from [5], where the case without constraints was considered.

Theorem 1 *Let $f : X \times Y \to [-\infty, +\infty]$ be an extended real-valued function and $K : X \rightrightarrows Y$ be a set-valued mapping which satisfy (A1)-(A4). Let $\varepsilon > 0$ and $x_0 \in X$ be such that $v(x_0) > sup_{x \in X} v(x) - \varepsilon$ and let $\delta > 0$ and $y_0 \in Kx_0$ be such that $f(x_0, y_0) < \inf_{y \in Kx_0} f(x_0, y) + \delta$. Then, there exist continuous bounded functions $q : X \to \mathbb{R}_+$ and $p : Y \to \mathbb{R}_+$, such that $q(x_0) = p(y_0) = 0$, $\|q\|_{X,\infty} \leq \varepsilon$, $\|p\|_{Y,\infty} \leq \delta$ and the supinf problem $\sup_{x \in X} \inf_{y \in Kx} \{f(x, y) - q(x) + p(y)\}$ has a solution at (x_0, y_0).*

If the function f and the set-valued mapping K satisfy the assumptions (A1)–(A4), and $\varepsilon > 0$ and $\delta > 0$ are arbitrary, then points x_0 and y_0 as in the statement above always exist.

Proof Consider the function $v(x) = \inf_{y \in Kx} f(x, y)$, $x \in X$. Since, under the assumptions of the theorem it is proper and bounded from above (assumption (A3)) and also upper semicontinuous (Lemma 3), then by Lemma 1, applied to the function $-v(\cdot)$, there is a continuous bounded function $q : X \to \mathbb{R}_+$, such that $q(x_0) = 0$, $\|q\|_{X,\infty} \leq \varepsilon$ and the function $\inf_{y \in K(\cdot)} f(\cdot, y) - q(\cdot)$ attains its maximum in X at x_0.

From assumption (A3) (the fact that v is proper and bounded above) it follows that $v(x_0) = \inf_{y \in Kx_0} f(x_0, y)$ is a finite number. Consequently, $f(x_0, \cdot)$ is a bounded below proper (and by (A4), also lower semicontinous) function with values in $\mathbb{R} \cup \{+\infty\}$, such that $Kx_0 \cap \mathrm{dom}\,(f(x_0, \cdot)) \neq \emptyset$. Then, since Kx_0 is also a closed set in Y, by Lemma 2 there exists a continuous bounded function $p : Y \to \mathbb{R}_+$, such that $p(y_0) = 0$, $\|p\|_{Y,\infty} \leq \delta$ and the function $f(x_0, \cdot) + p(\cdot)$ attains its minimum in Kx_0 at y_0. Moreover, $\|p\|_{Y,\infty} = f(x_0, y_0) - \inf_{y \in Kx_0} f(x_0, y) =: c \geq 0$.

With the so obtained functions q and p, take now some arbitrary $x \in X$ and fix it. Using the properties above we have:

$$\begin{aligned}\inf_{y\in Kx}\{f(x,y)-q(x)+p(y)\}-c &= \inf_{y\in Kx}\{f(x,y)+p(y)-c\}-q(x)\\ &\le \inf_{y\in Kx} f(x,y)-q(x)=v(x)-q(x)\\ &\le v(x_0)-q(x_0)=v(x_0),\end{aligned}$$

the last inequality being true because $v(\cdot)-q(\cdot)$ attains its maximum in X at x_0 and taking into account that $q(x_0)=0$.

Therefore, having in mind the above chain of inequalities and the definition of c, we obtain

$$\begin{aligned}\inf_{y\in Kx}\{f(x,y)-q(x)+p(y)\} &\le v(x_0)+c=\inf_{y\in Kx_0} f(x_0,y)+c=f(x_0,y_0)\\ &= f(x_0,y_0)+p(y_0)=\inf_{y\in Kx_0}\{f(x_0,y)+p(y)\}\\ &= \inf_{y\in Kx_0}\{f(x_0,y)-q(x_0)+p(y)\}.\end{aligned}$$

In the above we used the fact that the function $f(x_0,\cdot)+p(\cdot)$ attains its minimum on Kx_0 at y_0.

Combining the above chain of inequalities we conclude that (x_0,y_0) is a solution to the supinf problem generated by the function $f(x,y)-q(x)+p(y)$, $(x,y)\in X\times Y$.

In the following theorem (and in the sequel) on $C(X)\times C(Y)$ we consider the usual product topology, generated by the norms in $C(X)$ and $C(Y)$ respectively. It is easily seen that if a function $f: X\times Y\to[-\infty,+\infty]$ and a set-valued mapping $K: X\rightrightarrows Y$ satisfy the assumptions (A1)–(A4) and g is a continuous bounded function on $X\times Y$, then the function $f+g$ and the mapping K also satisfy (A1)–(A4). Therefore, the next theorem is an easy consequence of Theorem 1, having in mind the latter remark and the properties of the sup-norms.

Theorem 2 *Let $f: X\times Y\to[-\infty,+\infty]$ and $K: X\rightrightarrows Y$ satisfy the assumptions (A1)–(A4). Then, we have the following:*

(a) *The set $\{(q,p)\in C(X)\times C(Y)$: the function $f(x,y)+q(x)+p(y)$, $(x,y)\in X\times Y$, has a solution for the supinf problem (P)$\}$ is a dense subset of $C(X)\times C(Y)$;*
(b) *The set $\{u\in C(X\times Y)$: the function $f(x,y)+u(x,y)$, $(x,y)\in X\times Y$, has a solution for the supinf problem (P) $\}$ is a dense subset of $(C(X\times Y),\|\cdot\|_{X\times Y,\infty})$.*

The assertions from (a) and (b) above constitute "dense" variational principles for supinf problems with constraints.

4 Well-posedness of supinf problems

In this section we will give a characterization of the well-posedness of supinf problems with constraints. But before that, we give a result concerning the sup-well-posedness, defined in the Introduction. Recall that, given the objective function $f: X\times Y\to[-\infty,+\infty]$ and the constraint mapping $K: X\rightrightarrows Y$, the sup-well posedness of the corresponding supinf problem (P) is simply the well-posedness of maximization problem determined by the function $v(x)=\inf_{y\in Kx} f(x,y)$, $x\in X$. The following proposition is proved as Proposition 2.10 from [5], using Theorem 1, and this is why the proof is omitted. It shows that in some cases (for example, in metric spaces) we can obtain perturbations such that the corresponding perturbed supinf problem not only has a solution but it is also sup-well-posed.

Proposition 1 *Let $f : X \times Y \to [-\infty, +\infty]$ and $K : X \rightrightarrows Y$ satisfy the assumptions (A1)–(A4). Let $\varepsilon > 0$ and $x_0 \in X$ be such that $v(x_0) > sup_{x\in X} v(x) - \varepsilon$ and let $\delta > 0$ and $y_0 \in Kx_0$ be such that $f(x_0, y_0) < \inf_{y\in Kx_0} f(x_0, y) + \delta$. Suppose that x_0 has a countable local base in X. Then, there exist continuous bounded functions $q : X \to \mathbb{R}_+$ and $p : Y \to \mathbb{R}_+$, such that $q(x_0) = p(y_0) = 0$, $\|q\|_{X,\infty} \leq \varepsilon$, $\|p\|_{Y,\infty} \leq \delta$, the supinf problem $\sup_{x\in X} \inf_{y\in Kx}\{f(x, y) - q(x) + p(y)\}$ has a solution at (x_0, y_0) and the latter problem is also sup-well-posed with unique sup-solution x_0.*

Further, we will be interested in investigating the stronger notion of well-posedness of the supinf problems, defined again in the Introduction. To this end, let $S_f : C(X) \times C(Y) \rightrightarrows X \times Y$ be the set-valued mapping which assigns to every couple of functions $(q, p) \in C(X) \times C(Y)$ the (possibly empty) solution set to the problem $\sup_{x\in X} \inf_{y\in Kx}\{f(x, y) + q(x) + p(y)\}$.

Theorem 3 *Let $f : X \times Y \to [-\infty, +\infty]$ and $K : X \rightrightarrows Y$ satisfy the assumptions (A1)–(A4). Then the mapping S_f is single-valued and upper semicontinuous at $(q, p) \in C(X)\times C(Y)$ if and only if the supinf problem (P) for the function $f(x, y)+ q(x) + p(y)$, $(x, y) \in X \times Y$, and the mapping K is well-posed.*

Proof Suppose that S_f is upper semicontinuous and single-valued at $(q, p) \in C(X) \times C(Y)$. Let (x_0, y_0) be the unique solution to the corresponding supinf problem, that is $S_f(q, p) = \{(x_0, y_0)\}$. Let $((x_n, y_n))_n \in X \times Y$ be an optimizing sequence for the supinf problem generated by $f(x, y)+q(x)+p(y)$, $(x, y) \in X\times Y$. According to the definition of such a sequence we have (with a slight abuse of notation we use the symbol v_f for the value of the supinf problem generated by the function $f(x, y) + q(x) + p(y)$, $(x, y) \in X \times Y$):

1. $y_n \in Kx_n$ for each n;
2. $v(x_n) = \inf_{y\in Kx_n}\{f(x_n, y) + q(x_n) + p(y)\} \to v_f = \sup_{x\in X} \inf_{y\in Kx}\{f(x, y) + q(x) + p(y)\}$;
3. $f(x_n, y_n) + q(x_n) + p(y_n) \to v_f$.

Suppose that $((x_n, y_n))_n$ does not converge to (x_0, y_0). By passing to a subsequence, we may assume that there exist open neighbourhoods U of x_0 and V of y_0 such that $(x_n, y_n) \notin U \times V$ for every n.

From the upper semicontinuity of S_f at (q, p) there exists $\varepsilon > 0$, such that $||q' - q||_{X,\infty} < \varepsilon$, $||p' - p||_{Y,\infty} < \varepsilon$, $q' \in C(X)$ and $p' \in C(Y)$, implies $S_f(q', p') \subset U \times V$.

Let n be so large that

$$v_f - v(x_n) < \varepsilon/2$$

and

$$|v_f - f(x_n, y_n) - q(x_n) - p(y_n)| < \varepsilon/2$$

Then, in particular, combining the above two inequalities, for such a large n, we have

$$f(x_n, y_n) + q(x_n) + p(y_n) < v(x_n) + \varepsilon = \inf_{y\in K_{x_n}} \{f(x_n, y) + q(x_n) + p(y)\} + \varepsilon.$$

Fix such a large n which satisfies the last inequalities and apply Theorem 1 for the function $f(x, y) + q(x) + p(y)$, $(x, y) \in X \times Y$ (the latter function obviously satisfies (A1)–(A4) together with the mapping K) the point x_n with $\varepsilon/2$ (we have $v(x_n) > \sup_{x\in X} v(x) - \varepsilon/2$), and the point y_n with ε ($f(x_n, y_n) + q(x_n) + p(y_n) < \inf_{y\in Kx_n}\{f(x_n, y) + q(x_n) + p(y)\} + \varepsilon$). Then, there are functions $q_n \in C(X)$ and $p_n \in C(Y)$, such that $\|q_n\|_{X,\infty} \leq \varepsilon/2$, $\|p_n\|_{Y,\infty} \leq \varepsilon$ and (x_n, y_n) is a solution to the supinf problem for the function $f(x, y) + q(x) - q_n(x) + p(y) + p_n(y)$, $(x, y) \in X\times Y$. But $\|q-q_n-q\|_{X\infty} < \varepsilon/2$, $\|p+p_n-p\| < \varepsilon$ and $(x_n, y_n) \in S_f(q-q_n, p+p_n)$, which contradicts $(x_n, y_n) \notin U \times V$.

Conversely, suppose that the supinf problem for the function $f(x, y) + q(x) + p(y)$, $(x, y) \in X \times Y$, $(q, p) \in C(X)\times C(Y)$, and the mapping K, is well-posed with unique solution (x_0, y_0). Hence $S_f(q, p) = \{(x_0, y_0)\}$. Suppose that S_f is not usc at (q, p). Then, there exist open neighbourhoods U of x_0 and V of y_0, such that for every $n \geq 1$ there are $q_n \in C(X)$ and $p_n \in C(Y)$ such that $\|q_n - q\| < 1/n$, $\|p_n - p\| < 1/n$ and $S_f(q_n, p_n)$ is not included in $U \times V$. The latter implies that for every $n \geq 1$ there exists $(x_n, y_n) \in S_f(q_n, p_n) \setminus (U \times V)$.

Since for each $n \geq 1$ the couple (x_n, y_n) is a solution for the supinf problem for the function $f(x, y)+q_n(x)+p_n(y)$, $(x, y) \in X\times Y$, and the mapping $K : X \rightrightarrows Y$, then for every $n \geq 1$ we have $y_n \in Kx_n$ and :

$$\begin{aligned} f(x_n, y_n) + q_n(x_n) + p_n(y_n) &= \inf_{y\in Kx_n}\{f(x_n, y) + q_n(x_n) + p_n(y)\} \\ &= \sup_{x\in X} \inf_{y\in Kx}\{f(x, y) + q_n(x) + p_n(y)\}. \end{aligned}$$

For brevity, for each $n \geq 1$, denote by α_n, $v_n(x_n)$ and v_n the three (equal) expressions in the last chain of equalities. The fact that q_n and p_n converge uniformly to q and p, respectively (in the corresponding spaces), easily implies that v_n converges to $v_f = \sup_{x\in X} \inf_{y\in Kx}\{f(x, y) + q(x) + p(y)\}$, $v_n(x_n)$ and $v(x_n) = \inf_{y\in Kx_n}\{f(x_n, y) + q(x_n) + p(y)\}$ are close eventually, and α_n and $f(x_n, y_n) + q(x_n) + p(y_n)$ are also close eventually. All this together with the above equalities shows that $((x_n, y_n))_n$ is an optimizing sequence for the supinf problem determined by the function $f(x, y)+q(x)+p(y)$, $(x, y) \in X\times Y$. And this is a contradiction with the well-posedenss of the supinf problem, since $((x_n, y_n))_n$ does not converge to (x_0, y_0). This completes the proof of the theorem.

The reader should observe that, in the above proof, apart from the classical properties of the uniform convergence, what we have used concerning the assumptions imposed in order to apply Theorem 1, is that, if $f : X \times Y \to [-\infty, +\infty]$ and the mapping $K : X \rightrightarrows Y$ satisfy (A1)–(A4) and $(q, p) \in C(X) \times C(Y)$, then $f(x, y) + q(x) + p(y)$, $(x, y) \in X \times Y$, also satisfies the assumptions (A1)–(A4) for the mapping K. Therefore, the following theorem is proved exactly as the theorem above, having in mind our remark before Theorem 2, namely, that if we have a function f and K which satisfy (A1)–(A4), and $u \in C(X \times Y)$, then the function $f + u$ also satisfies (A1)–(A4) for the mapping K.

Let us consider the mapping $\tilde{S}_f : (C(X \times Y), \|\cdot\|_{X\times Y,\infty}) \rightrightarrows X \times Y$ which assigns to each $u \in C(X \times Y)$ the (possibly empty) solution set to the supinf problem (P) generated by the function $f + u$ and the mapping K. Then,

Theorem 4 *Let $f : X\times Y \to [-\infty, +\infty]$ and $K : X \rightrightarrows Y$ satisfy the assumptions (A1)–(A4). Then the mapping $\tilde{S}_f : (C(X \times Y), \|\cdot\|_{X\times Y,\infty}) \rightrightarrows X \times Y$ is single-valued and upper semicontinuous at $u \in C(X\times Y)$, if and only if the supinf problem (P) for the function $f + u$ and the mapping K is well-posed.*

The latter result is a generalization of Proposition 4.8 from [3], where it was proved for the case $f \equiv 0$.

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